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Entanglement in a squeezed two-level atom

Shigeru Furuichi[1] **and Mahmoud Abdel-Aty**[2]

[1] Science University of Tokyo in Yamaguchi, Onoda City, Yamaguchi, 756-0884, Japan
[2] Mathematics Department, Faculty of Science, South Valley University, 82524 Sohag, Egypt

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Abstract

In a previous paper, we adopted the method using quantum mutual entropy to measure the degree of entanglement in the time development of the Jaynes–Cummings model. In this paper, we formulate the entanglement in the time development of the Jaynes–Cummings model with squeezed states, and then show that the entanglement can be controlled by means of squeezing.

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1. Introduction

Recently, it has become known that a quantum entangled state plays an important role in such fields of quantum information theory as quantum teleportation and quantum computation. The research on quantifying entangled states has been done by several measures [1, 2].

If we want to quantify entangled states, we should know whether they are pure states or mixed states. That is, if the entangled states are pure states, then it is well known that it is sufficient to use von Neumann entropy [3] for the reduced states [1, 2], because, for pure states, it has a unique measure. However, for mixed states, it does not have a unique measure. Therefore we need a proper measure of entanglement for mixed states. The degree of entanglement for mixed states has been studied by some entropic measures such as entanglement of formation [1] and quantum relative entropy [2] and so on. Vedral *et al* [2] defined the degree of the entangled states σ as a minimum distance between all disentangled states $\rho \in \mathcal{D}$ such that $E(\sigma) \equiv \min_{\rho\in\mathcal{D}} D(\sigma\|\rho)$ where D is any measure of distance between the two states σ and ρ. For an example, we can choose quantum relative entropy as D. Then,

$$E(\sigma) = \min_{\rho\in\mathcal{D}} S(\sigma|\rho) \tag{1}$$

where $S(\sigma|\rho) \equiv \operatorname{tr}\sigma(\log\sigma - \log\rho)$ is quantum relative entropy [4, 5]. Since this measure has to take a minimum over all disentangled states, it is difficult to calculate analytically for an actual model, such as the Jaynes–Cummings model, so we use the degree of entanglement due to mutual entropy [6], which we call DEM in the following, as defined below. Moreover, there has been no fixed definition of entanglement measure, though some measures have been defined other than the above measure defined in (1). So we can use the convenient measure we want, case-by-case.

Let σ be a state in $\mathfrak{S}_1 \otimes \mathfrak{S}_2$ and ρ_k are the marginal states in $\mathfrak{S}_k$ (i.e., $\mathrm{tr}_j\, \sigma = \rho_k (k \neq j)$). Then our degree of entanglement due to mutual entropy (DEM) is defined by:

$$I_\sigma(\rho_1, \rho_2) \equiv \mathrm{tr}\, \sigma(\log \sigma - \log \rho_1 \otimes \rho_2).$$

Note that the tensor product state $\rho_1 \otimes \rho_2$ is one of the disentangled states. This DEM represents the difference between the entangled states and disentangled states. This quantity is also applied to classify entanglement in [7]. If we treat only entangled pure states, it is sufficient to apply von Neumann entropy for the *reduced* states $\rho_k \equiv \mathrm{tr}_j \sigma \in \mathfrak{S}_k (k \neq j)$. Because we suppose that $\sigma \in \mathfrak{S}_1 \otimes \mathfrak{S}_2$ are entangled pure states, then the von Neumann entropy is equal to 0 ($S(\sigma) = 0$). Moreover, according to the following triangle inequality of Araki and Lieb [8]:

$$|S(\rho_1) - S(\rho_2)| \leqslant S(\sigma) \leqslant S(\rho_1) + S(\rho_2)$$

we have $S(\rho_1) = S(\rho_2)$. Thus we have

$$\begin{aligned} I_\sigma(\rho_1, \rho_2) &= \mathrm{tr}\, \sigma(\log \sigma - \log \rho_1 \otimes \rho_2) \\ &= S(\rho_1) + S(\rho_2) - S(\sigma). \\ &= 2S(\rho_1). \end{aligned} \tag{2}$$

Therefore, for entangled pure states, the DEM becomes twice the entropy of the induced marginal states. That is, if we want to know the degree of the entangled pure states, it is sufficient to use the *reduced* von Neumann entropy. However, in general the reduced von Neumann entropies for entangled mixed states are not always unique, namely, they depend on how the partial trace is taken. Therefore we need a unique measure for the entangled mixed states. Thus in this paper we apply the DEM, not the reduced von Neumann entropy, in order to measure the degree of entanglement for the entangled mixed states, because the DEM can measure the degree of entanglement directly without taking the partial trace. From (2), we also find that the entanglement degree in pure states is bigger than that in mixed states. In this short paper, we will formulate the entanglement degree in the Jaynes–Cummings model with squeezed states using DEM and then try to control it by means of the squeezing parameter *r*.

2. Atomic system

The quantum electrodynamical interaction of a single two-level atom with a single mode of an electromagnetic field is described by the well known Jaynes–Cummings model (JCM) [9]. The JCM is the simplest nontrivial model of two interacting fully quantum systems and has an exact solution. It also demonstrates some interesting phenomena such as collapses and revivals. It has been investigated in detail by many researchers from various points of view. For details, the reader may refer to the excellent reviews [10, 11]. The JCM is not only an important problem itself but also gives an excellent example of the so-called quantum open system problem [12], namely the interaction between a system and a reservoir.

So, we treated the JCM as a problem in non-equilibrium statistical mechanics and applied quantum mutual entropy [13] based on von Neumann entropy by finding the quantum mechanical channel [13] which expresses the state change of the atom on the JCM. This study was an attempt to obtain a new insight into the dynamical change of the state for the atom on the JCM by the quantum mutual entropy [14].

On the other hand, this model has one of the most interesting features which is the entanglement developed between the atom and the field during the interaction. There have been several approaches to analyse the time evolution in this model, for instance, von Neumann entropy and atomic inversion. Moreover, it is not suitable for the quantum mutual entropy which is mentioned above to measure the degree of entanglement. Because the quantum mutual entropy is defined for the quantum mechanical channel and, when we obtain it, we take the partial trace over one system, then the entanglement tends to loss. Therefore, in this short paper, we will apply the DEM to analyse the entanglement of the time development of the JCM, since the DEM can measure the degree of entangled states directly without taking the partial trace.

The resonant JCM Hamiltonian can be expressed by the rotating-wave approximation in the following form:

$$\begin{aligned} &H = H_A + H_F + H_I \\ &H_A = \frac{1}{2}\hbar\omega_0\sigma_z \qquad H_F = \hbar\omega_0 a^* a \\ &H_I = \hbar g(a\otimes\sigma^+ + a^*\otimes\sigma^-) \end{aligned}$$

where g is a coupling constant, $\sigma^{\pm}$ are the pseudo-spin matrices of a two-level atom, σ_z is the z-component of the Pauli spin matrix, and a (resp. a^*) is the annihilation (resp. creation) operator of a photon. In general, it is almost impossible to physically realize the pure states, so we suppose the initial states of the atom are the mixed states which are the more realistic representation of the states. We now suppose that the initial states of the atom are superposition states of the ground states and the excited states:

$$\rho = \lambda_0 E_0 + \lambda_1 E_1 \in \mathfrak{S}_A$$

where $E_0 = |1\rangle\langle 1|,\ E_1 = |2\rangle\langle 2|,\ \lambda_0 + \lambda_1 = 1$. This means that we have a thermalized atom where there is no coherence between the levels. Let the field initially be in squeezed states:

$$\omega = |\theta;\xi\rangle\langle\theta;\xi| \in \mathfrak{S}_F \qquad |\theta;\xi\rangle = \exp\left(-\frac{1}{2}|\beta|^2\right)\sum_l \frac{\beta^l}{\sqrt{l!}}|l;\xi\rangle$$

where $\beta = \mu\theta + \nu\theta^*,\ \xi = r\mathrm{e}^{\mathrm{i}\theta},\ \mu = \cosh r,\ \nu = \sinh r$ and r is often called a squeezing parameter. The continuous map $\mathcal{E}_t^*$, which is often called *lifting* [15], describing the time evolution between the atom and the field for the JCM, is defined by the unitary operator generated by H such that

$$\begin{aligned} &\mathcal{E}_t^* : \mathfrak{S}_A \to \mathfrak{S}_A \otimes \mathfrak{S}_F \\ &\mathcal{E}_t^*\rho = U_t(\rho\otimes\omega)U_t^* \\ &U_t \equiv \exp\left(-\mathrm{i}t\frac{H}{\hbar}\right). \end{aligned} \tag{3}$$

This unitary operator U_t is written as

$$U_t = \exp\left(-\frac{\mathrm{i}tH}{\hbar}\right) = \sum_{n=0}^{\infty}\sum_{j=0}^{1} E_{n,j}\left|\Phi_j^{(n)}\right\rangle\left\langle\Phi_j^{(n)}\right| \tag{4}$$

where $E_{n,j} = \exp[-\mathrm{i}t\{\omega_0(n+\frac{1}{2}) + (-1)^j\Omega_n\}]$ are the eigenvalues with $\Omega_n = g\sqrt{n+1}$, called the Rabi frequency, and $\Phi_j^{(n)}$ are the eigenvectors associated with $E_{n,j}$.

The transition probability for which the atom is initially prepared in the excited state and stays in the excited state after time t is given by

$$c(t) = |\langle n\otimes 2|\, U_t\, |n\otimes 2\rangle|^2 = \sum_n P(n)\cos^2\Omega_n t.$$

Also the transition probability for which the atom is initially prepared in the excited state and is at the grounded state after the time t is given by

$$s(t) = |\langle n+1 \otimes 1|U_t|n \otimes 2\rangle|^2 = \sum_n P(n) \sin^2 \Omega_n t.$$

We note here that $P(n)$ is formulated by

$$P(n) = \frac{1}{|\mu| n!} \left(\frac{|\nu|}{2|\mu|} \right)^n \left| H_n \left(\frac{\beta}{\sqrt{2\mu\nu}} \right) \right|^2 \exp \left(-|\beta|^2 + \frac{\nu}{2\mu}\beta^2 + \frac{\nu^*}{2\mu}\beta^{*2} \right) \quad (5)$$

where $\beta = \mu\theta + \nu\theta^*$ and $H_n(x)$ are Hermite polynomials.

3. Derivation of the DEM

In this section, we derive the DEM for a two-level atom with squeezed state. From (4) and (5), we obtain the lifting expression as follows;

$$\begin{aligned}
\mathcal{E}_t^* \rho = \sum_n & \left\{ \lambda_0 P(n+1) \sin^2 \Omega_n t + \lambda_1 P(n) \cos^2 \Omega_n t \right\} |2\rangle\langle 2| \otimes |n\rangle\langle n| \\
& + \frac{\mathrm{i}}{2} \sum_n \sin 2\Omega_n t (\lambda_1 P(n) - \lambda_0 P(n+1)) |2\rangle\langle 1| \otimes |n\rangle\langle n+1| \\
& + \frac{\mathrm{i}}{2} \sum_n \sin 2\Omega_n t (\lambda_0 P(n+1) - \lambda_1 P(n)) |1\rangle\langle 2| \otimes |n+1\rangle\langle n| \\
& + \sum_n \left\{ \lambda_0 P(n) \sin^2 \Omega_n t + \lambda_1 P(n+1) \cos^2 \Omega_n t \right\} |1\rangle\langle 1| \otimes |n+1\rangle\langle n+1|. \qquad (6)
\end{aligned}$$

According to [16], both atomic and field entropies are equal when the system is isolated; that is, the final states are the pure states. However, since we have $\mathcal{E}_t^* \rho \neq (\mathcal{E}_t^* \rho)^2$, the final states of the JCM are the entangled mixed states, so we should apply the DEM, not the reduced von Neumann entropy. Thus the DEM for a two-level atom with squeezed state is given by

$$\begin{aligned}
I_{\mathcal{E}_t^* \rho}\left(\rho_t^A, \rho_t^F\right) &= \operatorname{tr} \mathcal{E}_t^* \rho \left(\log \mathcal{E}_t^* \rho - \log \rho_t^A \otimes \rho_t^F \right) \\
&= -2(e_1(t) \log e_1(t) + e_4(t) \log e_4(t)) + \kappa_+(t) \log \kappa_+(t) + \kappa_-(t) \log \kappa_-(t) \qquad (7)
\end{aligned}$$

where

$$\begin{aligned}
\kappa_\pm(t) &= \frac{1}{2} \left\{ (e_1(t) + e_4(t)) \pm \sqrt{(e_1(t) + e_4(t))^2 - 4(e_1(t)e_4(t) - e_2(t)e_3(t))} \right\} \\
e_1(t) &= \sum_n \left\{ \lambda_0 P(n+1) \sin^2 \Omega_n t + \lambda_1 P(n) \cos^2 \Omega_n t \right\} \\
e_2(t) &= \frac{\mathrm{i}}{2} \sum_n \sin 2\Omega_n t (\lambda_1 P(n) - \lambda_0 P(n+1)) \\
e_3(t) &= \frac{\mathrm{i}}{2} \sum_n \sin 2\Omega_n t (\lambda_0 P(n+1) - \lambda_1 P(n)) \\
e_4(t) &= \sum_n \left\{ \lambda_0 P(n) \sin^2 \Omega_n t + \lambda_1 P(n+1) \cos^2 \Omega_n t \right\}.
\end{aligned}$$

4. Numerical computations

On the basis of the analytical solution presented in the previous section, in this section we examine the temporal evolution of the transition probability $c(t)$, the probability distribution and the degree of entanglement for different values of the squeezed parameter r.

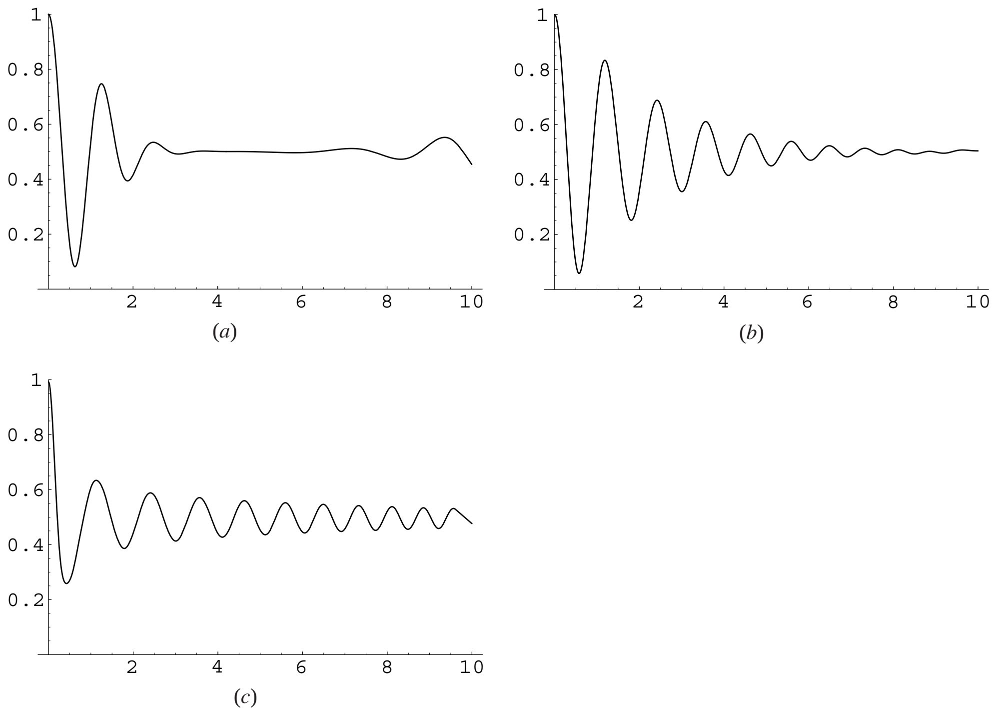

Figure 1. $c(t)$ for time t as $r = 0$ (a), $r = 1$ (b) and $r = 2$ (c).

We display the time evolution of the transition probability $c(t)$ in a two-level atom with squeezed states in figure 1, since this measure has often been used to analyse the time development of the JCM. Figure 1(a) is obtained by setting the squeezing parameter $r = 0$, namely a coherent state which is a special case of the squeezed states and the change of the figure is well known as a nature of the coherent states JCM [17]. To visualize the influence of the squeezed states in the transition probability $c(t)$ we set different values of the squeezing parameter r (see figures 1(b) and 1(c)), while all the other parameters are the same as in figure 1(a). From these figures, it is remarkable that the frequency of the oscillations gradually gets to increase with the gain of the squeezing parameter r. However, the size of the width of the amplitude in the three figures is not monotone for the gain of the squeezing parameter r.

These incomprehensible phenomena depend on the oscillation of the probability distribution of the squeezed state with increasing r [18], as also shown in figure 2, approximately described as a function of the photon number n in the case of the mean photon number $|\theta| = \sqrt{5}$. It is known that the source of the oscillation is caused by the interference [19]. The effect of the squeezing for a two-level atom is examined in [20].

We would like to know how this property of the squeezing influences the entanglement of the JCM. Figure 3 shows the three-dimensional plot of the DEM as a function of λ_1 and r when the time t is fixed at $t_r = 2\pi\theta/g$, which is often called the revival time. From this figure, we find that the DEM is not monotone for the squeezing parameter r due to non-monotonicity of the amplitude of the transition probability for the squeezing parameter r. The probability distribution functions in squeezed states oscillate as the squeezing parameter r (see figure 2), and it is known that this oscillation is caused by the interference in phase space [19]. From figure 3, we also find that the degree of entanglement in $r = 3$ is stronger than that in $r = 0$,

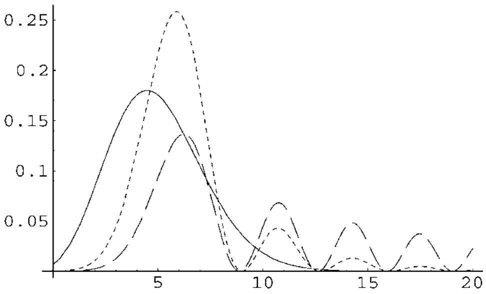

Figure 2. The probability distribution $P(n)$ for $r = 0$ (solid line), $r = 1$ (dotted line) and $r = 2$ (dashed line).

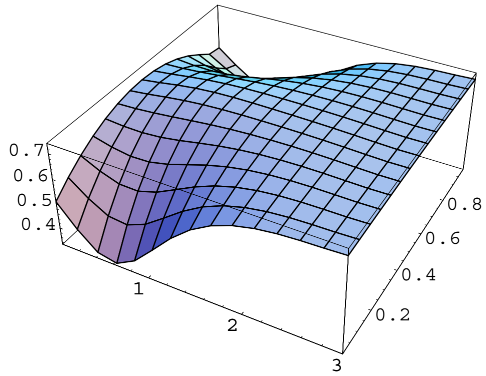

Figure 3. The DEM for λ_1 ($0 \leqslant \lambda_1 \leqslant 1$) and r ($0 \leqslant r \leqslant 3$) when the time t is fixed at $t_r = 2\pi\theta/g$.

which means that by squeezing we can obtain stronger entanglement than when we use the coherent states as the initial photon states; that is, it is possible to control the degree of the entanglement in the JCM by means of the squeezed states.

An interesting feature to observe in figure 3 is the symmetry around $\lambda_1 = 0.5$, which means that the DEM for the atom, whether in ground or excited states, has the same probability. We also find by using the initial mixed states of the atom that the DEM always takes the maximum value in $\lambda_1 = 0.5$ for any r and the minimum value in $\lambda_1 = 0$ or 1 for any r. This result shows that we obtain the maximum degree of entanglement in the JCM when we use the most mixed states $\rho = 0.5E_0 + 0.5E_1$ as the initial atomic state. This may be useful for the construction of a quantum computer.

5. Conclusion

We have studied the influence of the squeezed states on the degree of entanglement which is defined due to mutual entropy for a two-level atom. This shows that the degree of entanglement is very sensitive to the squeezing parameter. For small values of the squeezing parameter, a decrease of the degree of entanglement is shown, while for large values, an increase of the degree of entanglement is obtained. This is manifested in the degree of entanglement as it settles to a constant value for further increase of the squeezing parameter. This means that one can control the degree of entanglement by using the squeezing. We also found that the degree of entanglement in the JCM for any squeezing parameter r takes the maximum value, applying the mixed states to the initial atomic states. Moreover, we are interested in knowing if the DEM has an upper bound for squeezing parameter r. However, from figure 3 alone, we do not know whether the DEM has an upper bound or goes to infinity. This will remain as an ongoing problem.

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